Roth's theorem on arithemtic progressions in many variables

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The famous theorem of Roth (1953) states that if a set $A \subseteq \{1, \ldots, N\}$ does not contain any non-trivial 3 term arithmetic progression (a non-trivial solution to the equation $x + y = 2z$) then

$$|A| \leq C \frac{N}{\log \log N}$$

for some absolute constant C. This theorem became the focal point of the additive combinatorics, and many attempts to improve the above bound led to the development of interesting new techniques. Roth's theorem was refined by Heath-Brown (1987), Szemerédi (1990), Bourgain (1999, 2008), Sanders (2012) and very recently by Bloom. The current best upper bounds, due to Sanders and Bloom, give

$$|A| \leq \frac{N}{(\log N)^{1-o(1)}}.$$

On the other hand, Behrend (1946) constructed subsets of $\{1, \ldots, N\}$ of size

$$e^{-C\sqrt{\log N}} N,$$

$C > 0$, with no non-trivial progressions. This construction can be easily extended to a wider class of equations, in particular to equations of the form $x_1 + \cdots + x_k = ky$, $k \geq 2$.

We present the historical background of the problem, the techniques developed in this area and new tight results concerning the equation $x + y + z = 3w$.