Taylor's functional calculus and derived categories of Fréchet modules

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J. L. Taylor's functional calculus theorem (1970) asserts that every commuting n-tuple $T = (T_1, \ldots, T_n)$ of bounded linear operators on a Banach space E admits a holomorphic functional calculus on any neighborhood U of the joint spectrum $\sigma(T)$. This means that there exists a continuous homomorphism $\gamma : \mathscr{O}(U) \to \mathscr{B}(E)$ (where $\mathscr{O}(U)$ is the algebra of holomorphic functions on U and $\mathscr{B}(E)$ is the algebra of bounded linear operators on E) that takes the coordinates $z_1, \ldots, z_n$ to $T_1, \ldots, T_n$, respectively. The original Taylor's proof was quite involved. In 1972, Taylor developed a completely different and considerably shorter proof based on methods of homological algebra. Later it was simplified and generalized by M. Putinar (1980) to the case of Fréchet $\mathscr{O}(X)$-modules, where X is a finite-dimensional Stein space. The idea of Taylor-Putinar's construction is to establish an isomorphism between a Fréchet $\mathscr{O}(X)$-module M satisfying $\sigma(M) \subset U$ and the 0th cohomology of a certain double complex C of Fréchet $\mathscr{O}(U)$-modules. Unfortunately, C depends on the choice of a special cover of X by Stein open sets, and there seems to be no canonical way of associating C to M.

Our goal is to extend Taylor-Putinar's theorem to the setting of derived categories. We believe that this is exactly the environment in which Taylor-Putinar's theorem is most naturally formulated and proved. Given an object M of the derived category $\mathsf{D}^-(\mathscr{O}(X)\text{-}\underline{\mathsf{mod}})$ of Fréchet $\mathscr{O}(X)$-modules, we define the spectrum $\sigma(M) \subset X$, and we show that for every open set $U \subset X$ containing $\sigma(M)$ there is an isomorphism $M \cong \mathrm{R}\Gamma(U, \mathscr{O}_X) \mathbin{\widehat{\otimes}}^{\mathrm{L}}_{\mathscr{O}(X)} M$ in $\mathsf{D}^-(\mathscr{O}(X)\text{-}\underline{\mathsf{mod}})$. In the special case where M is a Fréchet $\mathscr{O}(X)$-module, this yields Taylor-Putinar's result. Moreover, we have $C = \mathrm{R}\Gamma(U, \mathscr{O}_X) \mathbin{\widehat{\otimes}}^{\mathrm{L}}_{\mathscr{O}(X)} M$, so C is natural in M when viewed as an object of the derived category.