# Reaching generalized critical values of a polynomial

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*The talk is based on the joint work with Krzysztof Kurdyka*

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Let $f\colon \mathbb{K}^n \to \mathbb{K}$ be a polynomial ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). Over forty years ago R. Thom proved that $f$ is a $C^\infty-$fibration outside a finite subset of the target, the smallest such a set is called *the bifurcation set of* $f$, we denote it by $B(f)$. In a natural way appears a fundamental question: how to determine the set $B(f)$.

Let us recall that in general the set $B(f)$ is bigger than $K_0(f)$ - the set of critical values of $f$. It contains also the set $B_\infty(f)$ of bifurcations points at infinity. Briefly speaking the set $B_\infty(f)$ consists of points at which $f$ is not a locally trivial fibration at infinity (i.e., outside a large ball). To control the set $B_\infty(f)$ one can use the set of *asymptotic critical values of* $f$

$$K_\infty(f) = \{y \in \mathbb{K} : \exists x_\nu \in \mathbb{K}^n,\ x_\nu \to \infty \ s.t.\ f(x_\nu) \to y \ and\ \|x_\nu\|\|df(x_\nu)\| \to 0\}.$$

If $c \notin K_\infty(f)$, then it is usual to say that $f$ satisfies *Malgrange's condition* at $c$. It is proved, that $B_\infty(f) \subset K_\infty(f)$. We call $K(f) = K_0(f) \cup K_\infty(f)$ *the set of generalized crtitical values of* $f$. Thus we have that in general $B(f) \subset K(f)$. In the case $\mathbb{K} = \mathbb{C}$ we gave in [1] an algorithm to compute the set $K(f)$. In the real case, that is for a given real polynomial $f : \mathbb{R}^n \to \mathbb{R}$ we can compute $K(f_\mathbb{C})$ the set of generalized critical values of $f_\mathbb{C}$ which stands for the complexification of $f$. However in general the set $K_\infty(f)$ of asymptotic critical values of $f$ may be smaller than $\mathbb{R} \cap K_\infty(f_\mathbb{C})$.

In the lecture we propose another approach to the computation of generalized critical values which works both in the complex and in the real case. The main new idea is to use a finite dimensional space of rational arcs along which we can reach all asymptotic critical values.

## References

[1] Z. Jelonek, K. Kurdyka, *On asymptotic critical values of a complex polynomial*, J. für die reine und angewandte Mathematik 565 (2003), 1–11.