Convexifying positive polynomials and s.o.s. approximation

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Important problems of real algebraic geometry are representations of non-negative polynomials on closed semialgebraic sets. Recall the 17th Hilbert problem (solved by E. Artin (1927)): if $f \in \mathbb{R}[x]$ is non-negative on $\mathbb{R}^n$, then $fh^2 = h_1^2 + \cdots + h_m^2$ for some $h, h_1, \ldots, h_m \in \mathbb{R}[x]$, $h \neq 0$, that is, f is a sum of squares of rational functions. If f is homogeneous and $f(x) > 0$ for $x \neq 0$, B. Reznick (1995) proved that the polynomial $(x_1^2 + \cdots + x_n^2)^N f(x)$ is a sum of even powers of linear functions provided $N \in \mathbb{Z}$ is sufficiently large.

Let $X \subset \mathbb{R}^n$ be a closed basic semialgebraic set defined by $g_1, \ldots, g_r \in \mathbb{R}[x]$, i.e., $X = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_r(x) \geq 0\}$. The preordering generated by $g_1, \ldots, g_r$ denoted by $T(g_1, \ldots, g_r)$ is defined to be the set of polynomials of the form $\sum_{e \in \{0,1\}^r} \sigma_e g_1^{e_1} \cdots g_r^{e_r}$, where $\sigma_e \in \sum \mathbb{R}[x]^2$ for $e \in \{0,1\}^r$ and $\sum \mathbb{R}[x]^2$ denotes the set of sums of squares (s.o.s.) of polynomials from $\mathbb{R}[x]$. Natural generalizations of the above theorem of Artin are the Stellensätze of J.-L. Krivine (1964), D. W. Dubois (1969), and J.-J. Risler (1970). When the set X is compact, a very important result was obtained by K.Schmüdgen (1991): every strictly positive polynomial f on X belongs to $T(g_1, \ldots, g_r)$. C. Berg, J. P. R. Christensen and P. Ressel (1976) and J. B. Lasserre and T. Netzer (2007) proved that any polynomial f which is non-negative on $[-1,1]^n$ can be approximated in the l_1-norm by sums of squares of polynomials. In this connection J. B. Lasserre (2008) obtained a result on approximation in the l_1-norm of convex polynomials provided that $g_1, \ldots, g_r$ are concave.

We show that a polynomial $f \in \mathbb{R}[x]$ is non-negative on the set X, if and only if f can be approximated uniformly on compact sets by polynomials of the form $\sigma_0 + \varphi(g_1) \cdot g_1 + \cdots + \varphi(g_r) \cdot g_r$, where $\sigma_0 \in \mathbb{R}[x]^2$ and $\varphi \in \mathbb{R}[t]^2$. Moreover, if X is a convex set such that $0 \notin X$, and d is a positive even number such that $d > \deg f$, then the above conditions are equivalent to: for any $a > 0$ there exists $N_0 \in \mathbb{N}$ such that for any integer $N \geq N_0$ the polynomial $\varphi_N(x) = (1 + |x|^2)^N (f(x) + a|x|^d)$ is a strictly convex function on X.

Additionally, we give necessary and sufficient conditions for the existence of an exponent $N \in \mathbb{N}$ such that $(1 + |x|^2)^N f(x)$ is a convex function on X.