Parameter-dependence of ODE's and regularity in the theory of infinite-dimensional Lie groups

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Let G be a Lie group modelled on a locally convex space, with identity element e and Lie algebra $\mathfrak{g}$. We say that G is C^0-*semiregular* if every continuous curve $\gamma\colon [0,1] \to \mathfrak{g}$ arises as the left logarithmic derivative of a (necessarily unique) C^1-curve $\eta = \eta_\gamma\colon [0,1] \to G$ with $\eta(0) = e$. If, moreover, the map

$$\mathrm{evol}\colon C([0,1],\mathfrak{g}) \to G, \quad \gamma \mapsto \eta_\gamma(1)$$

is smooth, then G is called C^0-*regular*. Thus, we are interested in the existence of solutions to certain initial value problems on a Lie group, and their dependence on parameters. I show that continuous dependence frequently entails smooth dependence. To this end, I first observe that evol is continuous if and only if it is continuous at 0. The main result then reads:

Theorem. *If G is C^0-semiregular, evol is continuous at* 0 *and the smooth homomorphisms from G to C^0-regular Lie groups separate points on G, then G is C^0-regular.*

As an application, consider a finite-dimensional Lie group H with compact Lie algebra $\mathfrak{h}$. As recently shown in a Master's thesis by Timm Pieper (Paderborn), there is a Lie group $C^\infty_{\mathcal{W}}(\mathbb{R}, H)$ of certain H-valued smooth maps on the line which is modelled on the weighted function space $C^\infty_{\mathcal{W}}(\mathbb{R}, \mathfrak{h})$, for the set $\mathcal{W}$ of all weight functions $f_a\colon \mathbb{R} \to]0,\infty[$, $f_a(t) := e^{-a|t|}$ with $a > 0$. I'll explain how the above theorem can be used to see that the Lie group $C^\infty_{\mathcal{W}}(\mathbb{R}, H)$ is C^0-regular.